Application of a Perturbation Method to Heat Flow Analysis in Materials Having Temperature-Dependent Properties

ULF OLSSON*
Svenska Flygmotor AB, Trollhattan, Sweden

Introduction

IN modern technology materials are often exposed to temperature ranges of such magnitudes that material property variations will be of great importance. The perturbation method or the method of successive approximations has proved itself to be a valuable tool in the analysis of the resulting nonlinear boundary value problems.[1,2] It is intended here to study the application of this method to heat flow problems from a more theoretical point of view.

The solutions are expanded in powers of one parameter common for the various material properties, thereby splitting up the original nonlinear problem into a set of interdependent linear ones. It is shown that the difficulties arising because of the products of derivatives in these equations may be released through a suitable transformation [Theorem 1, Eq. (10)]. In addition a method is given (Theorem 2), which makes it possible to predict the convergence of the series.

This method may also be used to obtain bounds on the solutions to the classical linear heat flow problem. It is a generalization of a method given by Boley[3] to include transient conditions, internal heat generation and convective heat transfer to the surface of the body. The principle is to obtain the temperature bounds starting out from assumed bounds on the prescribed heat flows. This is contrary to the method used by e.g., Appl and Hung,[4] who choose bounds on the temperature in such a way that the heat generated in the body is approximated as well as possible.

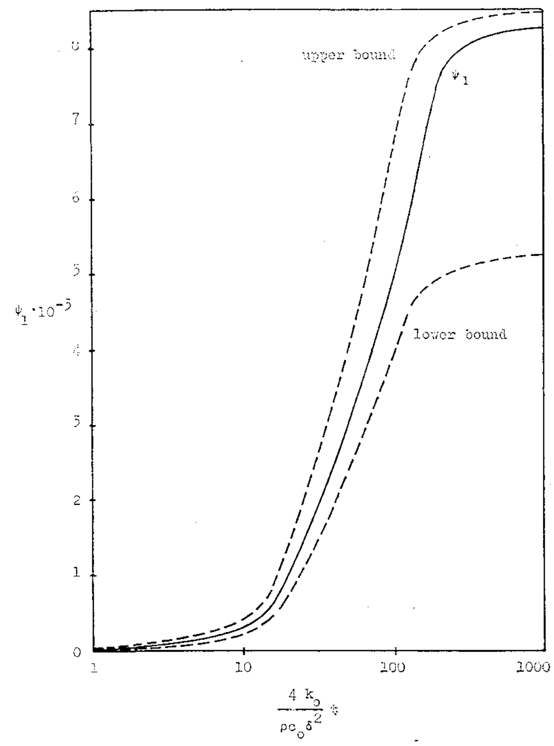

Fig. 1 The function Ψ_1 with upper and lower bounds.

Received May 25, 1970.
* Scientist, Research Department.

Theory

Consider a continuum of volume V, boundary B, and temperature T_s. At $t = 0$ this body is exposed to a heat load at the boundary and a time and space varying temperature field is generated, described by the following boundary value problem:

$$[k(\theta)\theta_{,j}]_{,j} + Q = \rho c(\theta)\partial\theta/\partial t \text{ in V} \quad (1)$$

$$-k(\theta)\theta_{,j}\nu_j - h(\theta)(\theta - \theta_a) = q_r \text{ on B} \quad (2)$$

$$\theta = T - T_s = 0 \text{ at } t = 0 \quad (3)$$

where k is the heat conductivity, c the specific heat, h the film heat-transfer coefficient, θ_a the ambient temperature, Q the heat generated per unit volume of the body, q_r the surface heat flux and ν the normal to the boundary, which is taken positive in the outward direction.

Theorem 1

Let the material properties k, c, and h be written as

$$f(\theta) = f_0 + \epsilon f_1\theta + \epsilon^2 f_2\theta^2 \quad (4)$$

where f stands for k, c, or h. Then using ϵ as a perturbation parameter, the nonlinear boundary value problem Eqs. (1–3) may be reduced to the following set of linear problems:

$$k_0\psi_{m,ii} + Q_m - \rho c_0(\partial\psi_m/\partial t) = 0 \text{ in V} \quad (5)$$

$$-k_0\psi_{m,i}\nu_i - h_0\psi_m = q_m \text{ on B} \quad (6)$$

$$\psi_m = 0 \text{ at } t = 0 \quad (7)$$

$$m = 0,1,2 \ldots$$

where

$$Q_m = Q\delta_{m0} - \frac{1}{2}\rho c_0\left(\frac{c_1}{c_0} - \frac{k_1}{k_0}\right)\frac{\partial\phi_{m-1}'}{\partial t} - \frac{1}{3}\rho c_0\left(\frac{c_2}{c_0} - \frac{k_2}{k_0}\right)\frac{\partial\phi_{m-2}''}{\partial t} \quad (8)$$

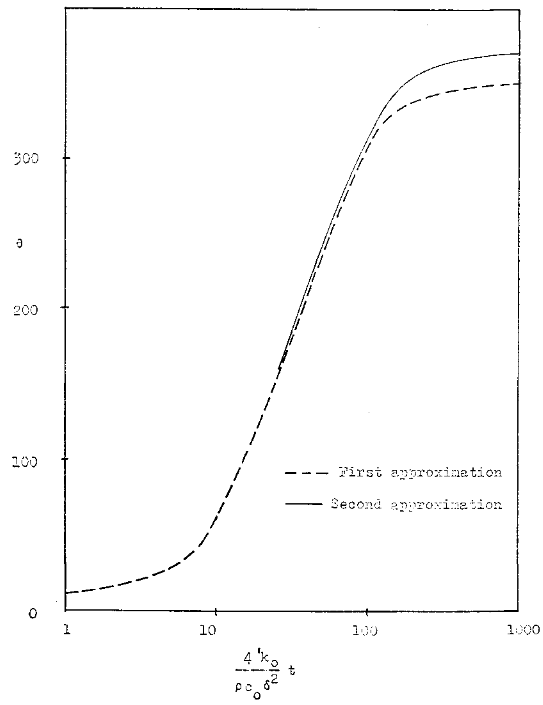

Fig. 2 First and second approximation of the temperature in the middle of the plate.

$$q_m = q_r\delta_{m0} - h_0\left(\frac{1}{2}\frac{k_1}{k_0} - \frac{h_1}{h_0}\right)\phi_{m-1}' - h_0\left(\frac{1}{3}\frac{k_2}{k_0} - \frac{h_2}{h_0}\right)\phi_{m-2}'' - h_0\theta_a - h_1\theta_a\theta_{m-1} - h_2\theta_a\theta_{m-2} \quad (9)$$

$$\theta_m = -\tfrac{1}{2}(k_1/k_0)\phi_{m-1}' - \tfrac{1}{3}(k_2/k_0)\phi_{m-2}'' + \psi_m \quad (10)$$

$$\phi_m' = \sum_{n=0}^{m} \theta_{m-n}\theta_n \quad (11)$$

$$\phi_m'' = \sum_{n=0}^{m} \phi_{m-n}'\theta_n \quad (12)$$

$$\theta = \sum_{m=0}^{\infty} \epsilon^m\theta_m \quad (13)$$

This theorem is proved through introducing Eqs. (4) and (13) into Eqs. (1–3) and singling out coefficients of ϵ, which give a set of linear equations, containing products of derivatives to the functions θ_m. These products are removed by introducing Eq. (10), which considerably simplifies the problem and leads to Eqs. (5–9), containing only time derivatives of the products of θ_m.

In order to predict the convergence of the series (13), consider the general heat flow problem:

$$k_0T_{,ii} + Q = \rho c_0(\partial T/\partial t) \text{ in V} \quad (14)$$

$$k_0T_{,i}\nu_i + h_0T + q = 0 \text{ on B} \quad (15)$$

$$T = 0 \text{ at } t = 0 \quad (16)$$

If $Q \geq 0$ and $q \leq 0$ then from energy considerations $T \geq 0$ and the following theorem can be stated:

Theorem 2

Let T' be a solution to the heat flow problem Eqs. (14–16) with Q' an arbitrary constant and $q' = 0$, let T'' be the corresponding solution with $Q'' = 0$ and q'' an arbitrary constant. If in Eqs. (14–16)

$$q_m \leq q \leq q_M \quad (17)$$

$$Q_m + \rho c_0\frac{T'}{Q'}\frac{\partial Q_m}{\partial t} + \rho c_0\frac{T''}{q''}\frac{\partial q_M}{\partial t} \leq Q \leq Q_M + \rho c_0\frac{T'}{Q'}\frac{\partial Q_M}{\partial t} + \rho c_0\frac{T''}{q''}\frac{\partial q_m}{\partial t} \quad (18)$$

then at all times $t \geq 0$

$$(Q_m/Q')T' + (q_M/q'')T'' \leq T \leq (Q_M/Q')T' + (q_m/q'')T'' \quad (19)$$

The theorem is proved through the introduction of the fictitious temperatures

$$T_M = (Q_M/Q')T' + (q_m/q'')T'' - T \quad (20)$$

$$T_m = (Q_m/Q')T' + (q_M/q'')T'' - T \quad (21)$$

into Eqs. (14–16). Using Eqs. (17) and (18) it follows that $T_M \geq 0$ and $T_m \leq 0$, which proves Eq. (19).

The theorem proved previously, giving bounds on the solution to the linear heat flow problem, can be used to predict the magnitude of the function ψ_{m+1} once ψ_m has been solved from Eqs. (5–7) and the corresponding heat flows Q_{m+1} and q_{m+1} from Eqs. (8) and (9).

Example

The theory outlined previously is applied to the problem of finding the transient temperature field in a convectively heated plate of inconel, having the following material properties, ($\theta = T - 273°$K): $\rho = 8250$ kg/m^3, $k = 14.7 + 0.015\ \theta$W/m°K, and $c = 430 + 0.265\ \theta$Ws/kg°K. The ambient temperature θ_a and the heat-transfer coefficient h are taken as $\theta_a = 700°$K and $h = 190 + 0.029\ \theta$W/m^2°C. The thickness of the plate is taken to be $\delta = 3$ mm with no heat production within it and no heat flux at the surface, i.e., $Q = q_r = 0$.

The solution θ_0 for temperature-independent material properties is derived first and introduced into Eqs. (8) and (9) to give the intermediate heat flows Q_1 and q_1. After having chosen bounds on these functions so that the relations (17) and (18) are satisfied, the corresponding bounds on the function ψ_1 are obtained from Eq. (19) once ψ_1' and ψ_1'' have been solved from Eqs. (14–16). Lastly the function ψ_1 itself is derived from Eqs. (5–7) and a second approximation, $m = 1$, of the temperature field obtained from Eqs. (10–13).

The calculations have been carried out for a chosen value of $\epsilon = 0.01$ and with the following bounds on the heat flows: $Q_m = 0$, $Q_M = Q_1$, $q_m = q_1$, and $q_M = -10^6(1 - e^{-0.035t})$. The found time histories of the function ψ_1 and the temperature field in the middle of the plate are shown in Figs. (1) and (2) below, the solutions of the linear differential equations being taken from Ref. 5.

References

[1] Trostel, R., "Stationäre Wärmespannungen mit temperaturabhängigem Stoffwerten," Ingenieur-Archiv, 1958, p. 416.

[2] Ismail, I. A. and Nowinski, J. L., "Thermoelastic Problems for Shells of Revolution Exhibiting Temperature-Dependent Properties," *Applied Sciences Research,* Vol. 14, No. 3, 1964/1965, p. 211.

[3] Boley, B. A. and Weiner, J. K., *Theory of Thermal Stresses,* Wiley, New York, 1960, Ch. 6, p. 188.

[4] Appl, F. C. and Hung, H. M., "A Principle for Convergent Upper and Lower Bounds," *International Journal of Mechanical Sciences,* Vol. 6, 1964, p. 381.

[5] Carslaw, H. C. and Jaeger, J. C., "Conduction of Heat in Solids," 2nd ed., Oxford University Press, Oxford, Cambridge, Mass., 1959, Chap. 3.

A Simple Linear Approximation for Perturbed Motion about Moderately Elliptic Orbits

Alan M. Schneider* and Howard M. Koble†
University of California at San Diego, La Jolla, Calif.

In this Note, the linearized perturbation equations for a circular reference trajectory are applied to elliptic reference trajectories of small eccentricity. The concept is appealing because the closed form solution for the circular reference is so much simpler than the solution for the elliptic case. A numerical example indicates that the accuracy of the circular reference solution is more than adequate for many practical applications. An analysis is outlined to extend the results of the numerical example.

The linear, constant-coefficient differential equations that describe the perturbed motion of a body P relative to a reference body O in a circular orbit are well known.[1–3] The solution to these equations, which for brevity will be called the circular solution, is usually written in terms of coordinates in a local vertical coordinate system centered at O. The circular solution, which contains sines and cosines and terms linear in time, is simple enough to be calculated by hand. It has di-

Received October 15, 1969; revision received June 24, 1970. This research was supported by NASA under Grant NGR-05-009-106.

* Professor of Aerospace Engineering, Department of the Aerospace and Mechanical Engineering Sciences. Associate Fellow AIAA.

† National Science Foundation Trainee, Department of the Aerospace and Mechanical Engineering Sciences.